Appendix A

Controller canonical form

Given the discrete time state space model determined by the system matrix $\mathbf{\Phi}$, its state vector $\mathbf{x}_b$, the control matrix $\mathbf{\Gamma}$ and the measurement matrix $\boldsymbol{C}$ as presented in Section 2.3.5, the basis transformation Eqn.(2.88) (see [Tolle, 1985])

$$\mathbf{x} = \mathbf{T}\,\mathbf{x}_b \quad \text{and} \quad \mathbf{x}_b = \mathbf{T}^{-1}\,\mathbf{x} \tag{A.1}$$

can be carried out using the transformation matrix

$$\mathbf{T} = \left[\mathbf{t}_1, \ldots, \mathbf{t}_N\right]^T, \tag{A.2}$$

with $N = 8$ determining the order of the system with 4 subsystems of order $\nu_i = 2$ each. The new system matrices can be calculated via the transformation matrix $\mathbf{T}$ as

$$\begin{aligned} \mathbf{A} &= \mathbf{T}\,\mathbf{\Phi}\,\mathbf{T}^{-1}, && \text{(A.3)} \\ \mathbf{B} &= \mathbf{T}\,\mathbf{\Gamma}, && \text{(A.4)} \\ \mathbf{C} &= \boldsymbol{C}\,\mathbf{T}^{-1}. && \text{(A.5)} \end{aligned}$$

The desired structure of the system matrix in controller canonical form is according to Eqn.(2.94) and Eqn.(2.95)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ -a_2^{(11)} & -a_1^{(11)} & \ldots & -a_2^{(14)} & -a_1^{(14)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_2^{(41)} & -a_1^{(41)} & \ldots & -a_2^{(44)} & -a_1^{(44)} \end{bmatrix}, \tag{A.6}$$

and the structure of the control matrix in controller canonical form according to Eqn.(2.96) and Eqn.(2.97)

$$\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_4] = \begin{bmatrix} 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ 0 & \ldots & 1 \end{bmatrix}. \tag{A.7}$$

Using the transformation matrix defined by Eqn.(A.2), equation $\mathbf{A}\,\mathbf{T} = \mathbf{T}\,\mathbf{\Phi}$ can be rewritten for 4 subsystems as

$$\begin{aligned}
\mathbf{t}_2^T &= \mathbf{t}_1^T\,\mathbf{\Phi} && \text{(A.8)}\\
-a_2^{(11)}\mathbf{t}_1^T - a_1^{(11)}\mathbf{t}_2^T \ldots - a_2^{(14)}\mathbf{t}_7^T - a_1^{(14)}\mathbf{t}_8^T &= \mathbf{t}_2^T\,\mathbf{\Phi} && \text{(A.9)}\\
\vdots &= \vdots \\
\mathbf{t}_8^T &= \mathbf{t}_7^T\,\mathbf{\Phi} && \text{(A.10)}\\
-a_2^{(41)}\mathbf{t}_1^T - a_1^{(41)}\mathbf{t}_2^T \ldots - a_2^{(44)}\mathbf{t}_7^T - a_1^{(44)}\mathbf{t}_8^T &= \mathbf{t}_8^T\,\mathbf{\Phi} && \text{(A.11)}
\end{aligned}$$

or for all 4 independent vectors $\mathbf{t}_1$, $\mathbf{t}_3$, $\mathbf{t}_5$ and $\mathbf{t}_7$ in matrix form

$$\begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_7 \end{bmatrix} \cdot \begin{bmatrix} a_2^{(11)} + a_1^{(11)}\mathbf{\Phi} + \mathbf{\Phi}^2 & \ldots & a_2^{(14)} + a_1^{(14)}\mathbf{\Phi} \\ \vdots & \ddots & \vdots \\ a_2^{(41)} + a_1^{(41)}\mathbf{\Phi} & \ldots & a_2^{(44)} + a_1^{(44)}\mathbf{\Phi} + \mathbf{\Phi}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \qquad \text{(A.12)}$$

Using the Hamilton-Caley theorem [Tolle, 1985] leaves the determinant of the upper matrix zero and herewith the vectors $\mathbf{t}_1$, $\mathbf{t}_3$, $\mathbf{t}_5$ and $\mathbf{t}_7$ arbitrary. Therefore, the transformation matrix results in

$$\mathbf{T} = \left[\mathbf{t}_1, \mathbf{\Phi}^T\mathbf{t}_1, \ldots, \mathbf{t}_7, \mathbf{\Phi}^T\mathbf{t}_7\right]^T. \qquad \text{(A.13)}$$

With $\mathbf{B} = \mathbf{T}\,\mathbf{\Gamma}$ and the decomposition of the control matrix

$$\mathbf{\Gamma} = [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_4] \qquad \text{(A.14)}$$

for all subsystems the following conditions hold

$$\mathbf{b}_1 = \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_1^T\,\mathbf{\Phi} \\ \vdots \\ \mathbf{t}_7^T \\ \mathbf{t}_7^T\,\mathbf{\Phi} \end{bmatrix} \boldsymbol{\gamma}_1, \quad \ldots, \quad \mathbf{b}_4 = \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_1^T\,\mathbf{\Phi} \\ \vdots \\ \mathbf{t}_7^T \\ \mathbf{t}_7^T\,\mathbf{\Phi} \end{bmatrix} \boldsymbol{\gamma}_4 \qquad \text{(A.15)}$$

or rearranged

$$\mathbf{b}_1 = \mathbf{t}_1^T \underbrace{\begin{bmatrix} \boldsymbol{\gamma}_1 \\ \mathbf{\Phi}\,\boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{\gamma}_4 \\ \mathbf{\Phi}\,\boldsymbol{\gamma}_4 \end{bmatrix}}_{\mathbf{T}_c}, \quad \ldots, \quad \mathbf{b}_4 = \mathbf{t}_4^T \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \mathbf{\Phi}\,\boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{\gamma}_4 \\ \mathbf{\Phi}\,\boldsymbol{\gamma}_4 \end{bmatrix}. \qquad \text{(A.16)}$$

If all states are controllable and thus the matrix $\mathbf{T}_c$ is regular, the vectors for the transformation can be calculated by

$$\begin{aligned}
\mathbf{t}_1 &= \left[\mathbf{T}_c^T\right]^{-1}\mathbf{b}_1, && \text{(A.17)}\\
\vdots &= \vdots \\
\mathbf{t}_4 &= \left[\mathbf{T}_c^T\right]^{-1}\mathbf{b}_4. && \text{(A.18)}
\end{aligned}$$