Appendix B

Derivation of matrices $\mathbf{M}_k$ and $\mathbf{V}_k$

In Subsection 3.1.3 the matrices $\mathbf{M}_k$ and $\mathbf{V}_k$ have been defined in order to compute the gradient matrix $\boldsymbol{\Psi}$ recursively. A derivation of those matrices is given in the following.

The predictor based on the innovations model is introduced by Eqn.(3.3) and Eqn.(3.4) in Chapter 3 as

$$\begin{aligned} \hat{\mathbf{x}}(k+1,\mathbf{p}) &= \mathbf{A}(\mathbf{p})\,\hat{\mathbf{x}}(k,\mathbf{p}) + \mathbf{B}\,\mathbf{u}(k) + \mathbf{K}(\mathbf{p})\,\boldsymbol{\varepsilon}(k), && \text{(B.1)} \\ \hat{\mathbf{y}}(k) &= \mathbf{C}(\mathbf{p})\,\hat{\mathbf{x}}(k,\mathbf{p}), && \text{(B.2)} \end{aligned}$$

with its system matrices in controller canonical form as described in Chapter 2 and Appendix A as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ -a_2^{(11)} & -a_1^{(11)} & \dots & -a_2^{(14)} & -a_1^{(14)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_2^{(41)} & -a_1^{(41)} & \dots & -a_2^{(44)} & -a_1^{(44)} \end{bmatrix}, \qquad \text{(B.3)}$$

$$\mathbf{B} = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}, \qquad \text{(B.4)}$$

$$\mathbf{C} = \begin{bmatrix} c_2^{(11)} + c_1^{(11)} & \dots & c_2^{(14)} + c_1^{(14)} \\ \vdots & \ddots & \vdots \\ c_2^{(41)} + c_1^{(41)} & \dots & c_2^{(44)} + c_1^{(44)} \end{bmatrix}, \qquad \text{(B.5)}$$

and the Kalman matrix

$$\mathbf{K}(z) = \begin{bmatrix} k_2^{(11)} & \dots & k_2^{(14)} \\ k_1^{(11)} & \dots & k_1^{(14)} \\ \vdots & \ddots & \vdots \\ k_2^{(41)} & \dots & k_2^{(44)} \\ k_1^{(41)} & \dots & k_1^{(44)} \end{bmatrix}. \qquad \text{(B.6)}$$

This model yields 96 parameters to be estimated which are gathered within the parameter vector

$$\begin{aligned}\mathbf{p} = [\quad & -a_2^{(11)}, -a_1^{(11)}, \dots, -a_2^{(14)}, -a_1^{(14)}, \dots, \\ & -a_2^{(41)}, -a_1^{(41)}, \dots, -a_2^{(44)}, -a_1^{(44)}, \\ & c_2^{(11)}, c_1^{(11)}, \dots, c_2^{(14)}, c_1^{(14)}, \dots, \\ & c_2^{(41)}, c_1^{(41)}, \dots, c_2^{(44)}, c_1^{(44)}, \\ & k_2^{(11)}, k_1^{(11)}, \dots, k_2^{(14)}, k_1^{(14)}, \dots, \\ & k_2^{(41)}, k_1^{(41)}, \dots, k_2^{(44)}, k_1^{(44)} \quad].\end{aligned} \tag{B.7}$$

The parameter vector simply contains each second row of the system matrix $\mathbf{A}$ i.e. only $\frac{n_s^2}{2} = 32$ parameters, and all rows of the measurement matrix $\mathbf{C}$ and the Kalman matrix $\mathbf{K}$, both having no particular structure.

B.1 Derivation of the matrix $\mathbf{M}_k$

With the definition of the parameter vector the partial derivative $\mathbf{M}_k$ defined in Eqn.(3.38) is

$$\mathbf{M}_k(\hat{\mathbf{p}}, \mathbf{x}, \boldsymbol{\varepsilon}) = \frac{\partial}{\partial \mathbf{p}} \hat{\mathbf{x}}(k+1)\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = \frac{\partial}{\partial \mathbf{p}} \Big[\mathbf{A}(\mathbf{p})\,\mathbf{x} + \mathbf{B}\,\mathbf{u} + \mathbf{K}(\mathbf{p})\,\boldsymbol{\varepsilon}\Big]\Bigg|_{\mathbf{p}=\hat{\mathbf{p}}}. \tag{B.8}$$

Using the innovations model yields

$$\begin{aligned}\hat{x}_1(k+1) &= \hat{x}_2(k) + k_2^{(11)}\varepsilon_1(k) + \dots + k_2^{(14)}\varepsilon_4(k), \\ \hat{x}_2(k+1) &= -a_2^{(11)}\hat{x}_1(k) - a_1^{(11)}\hat{x}_2(k) \dots \\ &\quad -a_2^{(14)}\hat{x}_7(k) - a_1^{(14)}\hat{x}_8(k) \\ &\quad +k_1^{(11)}\varepsilon_1(k) + \dots + k_1^{(14)}\varepsilon_4(k), \\ &\vdots \\ \hat{x}_7(k+1) &= \hat{x}_8(k) + k_2^{(41)}\varepsilon_1(k) + \dots + k_2^{(44)}\varepsilon_4(k), \\ \hat{x}_8(k+1) &= -a_2^{(41)}\hat{x}_1(k) - a_1^{(41)}\hat{x}_2(k) \dots \\ &\quad -a_2^{(44)}\hat{x}_7(k) - a_1^{(44)}\hat{x}_8(k) \\ &\quad +k_1^{(41)}\varepsilon_1(k) \dots + k_1^{(44)}\varepsilon_4(k).\end{aligned} \tag{B.9}$$

Differentiating the above equations with respect to the parameter vector $\mathbf{p}$ results in the matrix

$$\mathbf{M}_k = \frac{\partial}{\partial \mathbf{p}} \hat{\mathbf{x}}(k+1)\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = \left[\mathbf{M}_x, \mathbf{0}^{8\times 32}, \mathbf{M}_\varepsilon\right] \tag{B.10}$$

with the 8×32-dimensional matrices

$$\mathbf{M}_x = \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} \\ \hat{\mathbf{x}}^T(k) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \dots & \hat{\mathbf{x}}^T(k) \end{bmatrix} \quad \text{and} \tag{B.11}$$

$$\mathbf{M}_\varepsilon = \begin{bmatrix} \boldsymbol{\varepsilon}(k) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}(k) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\varepsilon}(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\varepsilon}(k) \end{bmatrix}. \tag{B.12}$$

Note that the matrix $\mathbf{M}_x$ does not depend on $\mathbf{u}(k)$, because the model is used in controller canonical form.

B.2 Derivation of the matrix $\mathbf{V}_k$

The partial derivative of the expected output, $\mathbf{V}_k$, defined in Eqn.(3.31) is

$$\mathbf{V}_k(\hat{\mathbf{p}}, \mathbf{x}) = \frac{\partial}{\partial \mathbf{p}} \hat{\mathbf{y}}(k) \Big|_{\mathbf{p}=\hat{\mathbf{p}}} = \frac{\partial}{\partial \mathbf{p}} \Big[\mathbf{C}(\mathbf{p})\, \mathbf{x} \Big] \Bigg|_{\mathbf{p}=\hat{\mathbf{p}}}. \tag{B.13}$$

Using the innovations model yields

$$\begin{aligned} \hat{y}_1(k) &= c_2^{(11)} \hat{x}_1(k) + c_1^{(11)} \hat{x}_2(k) + \dots \\ & \quad c_2^{(14)} \hat{x}_7(k) + c_1^{(14)} \hat{x}_8(k), \\ \vdots \quad & \quad \vdots \\ \hat{y}_4(k) &= c_2^{(41)} \hat{x}_1(k) + c_1^{(41)} \hat{x}_2(k) + \dots \\ & \quad c_2^{(44)} \hat{x}_7(k) + c_1^{(44)} \hat{x}_8(k). \end{aligned} \tag{B.14}$$

Differentiating the above equations with respect to the parameter vector $\mathbf{p}$ results in the matrix

$$\mathbf{V}_k = \left[\mathbf{0}^{4 \times 32}, \mathbf{V}_x, \mathbf{0}^{4 \times 32} \right], \tag{B.15}$$

with

$$\mathbf{V}_x = \begin{bmatrix} \hat{\mathbf{x}}^T(k) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \hat{\mathbf{x}}^T(k) \end{bmatrix} \tag{B.16}$$

The matrices $\mathbf{M}_k$ and $\mathbf{V}_k$ are static in their structure, but have to be updated with the estimated states $\hat{\mathbf{x}}^T(k)$ and the prediction error $\boldsymbol{\varepsilon}(k)$ each sample interval.