Appendix C

Derivation of PI-controller

In Section 3.2 several derivations are implied which are presented in the following sections.

C.1 Closed loop transfer function

The output of the system controlled by a state feedback controller is according to Eqn.(3.83)

$$\mathbf{Y}(z) = \mathbf{G}_{cl}(z)\, \mathbf{U}_I(z) \tag{C.1}$$

with the transfer function matrix for the closed loop with simple state feedback

$$\mathbf{G}_{cl}(z) = \mathbf{C}\left(z\,\mathbf{I} - \mathbf{A} + \mathbf{B}\,\mathbf{K}_x\right)^{-1}\mathbf{B}. \tag{C.2}$$

Since the model is used in controller canonical form, the matrix $\mathbf{B}$ consists mainly of zeros and ones (see Appendix A). Thus, the the closed loop transfer function $\mathbf{G}_{cl}$ can be written with the definitions for the state space controller defined by Eqn.(3.70) and Eqn.(3.71) as

$$\mathbf{G}_{cl} = \begin{bmatrix} \frac{c_2^{(11)}+c_1^{(11)}z}{z^2+p_1^{(11)}z+p_2^{(11)}} & \cdots & \frac{c_2^{(14)}+c_1^{(14)}z}{z^2+p_1^{(11)}z+p_2^{(11)}} \\ \vdots & \ddots & \vdots \\ \frac{c_2^{(41)}+c_1^{(41)}z}{z^2+p_1^{(44)}z+p_2^{(44)}} & \cdots & \frac{c_2^{(44)}+c_1^{(44)}z}{z^2+p_1^{(44)}z+p_2^{(44)}} \end{bmatrix}. \tag{C.3}$$

This transfer function matrix can be put as a product of two transfer function matrices such that

$$\mathbf{G}_{cl} = \mathbf{N}(z)\,\mathbf{D}^{-1}(z) \tag{C.4}$$

with matrix $\mathbf{D}(z)$ as a diagonal matrix

$$\mathbf{D}(z) = \begin{bmatrix} z^2 + p_1^{(11)}z + p_2^{(11)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & z^2 + p_1^{(44)}z + p_2^{(44)} \end{bmatrix}, \tag{C.5}$$

and $\mathbf{N}(z)$ depending on the entries of matrix $\mathbf{C}$ as

$$\mathbf{N}(z) = \begin{bmatrix} c_2^{(11)} + c_1^{(11)}\, z & \dots & c_2^{(14)} + c_1^{(14)}\, z \\ \vdots & \ddots & \vdots \\ c_2^{(41)} + c_1^{(41)}\, z & \dots & c_2^{(44)} + c_1^{(44)}\, z \end{bmatrix}. \tag{C.6}$$

Using this convention, the output of the system controlled by a state feedback controller is

$$\mathbf{Y}(z) = \mathbf{N}(z)\, \mathbf{D}^{-1}(z)\, \mathbf{U}_I(z) \tag{C.7}$$

and with

$$\mathbf{U}_I(z) = \frac{1}{z-1}\mathbf{K}_I\, \mathbf{E}(z), \tag{C.8}$$

$$\mathbf{E}(z) = \mathbf{W}(z) - \mathbf{Y}(z), \tag{C.9}$$

the output finally depends only on the set-point error with

$$\mathbf{Y}(z) = \mathbf{N}(z)\, \mathbf{D}^{-1}(z)\, \frac{1}{z-1}\mathbf{K}_I\, \big(\mathbf{W}(z) - \mathbf{Y}(z)\big). \tag{C.10}$$

Solving Eqn.(C.10) for $\mathbf{Y}(z)$ yields

$$\mathbf{Y}(z) = \mathbf{G}_{cl_I}(z)\, \mathbf{W}(z) \tag{C.11}$$

with the transfer function matrix

$$\mathbf{G}_{cl_I}(z) = \Big(\mathbf{I} + \mathbf{N}\, \mathbf{D}^{-1}\, \frac{1}{z-1}\mathbf{K}_I\Big)^{-1}\, \mathbf{N}\, \mathbf{D}^{-1}\, \frac{1}{z-1}\mathbf{K}_I \tag{C.12}$$

The detailed derivation necessary in order to gain Eqn.(3.89) is presented in the following section.

C.2 Simplification of closed loop transfer function

Equation Eqn.(C.12) or Eqn.(3.89) can be simplified using the matrix inversion lemma with general matrices $\mathbf{X}$ and $\mathbf{Y}$

$$\Big(\mathbf{X}\, \mathbf{Y}\Big)^{-1} = \mathbf{Y}^{-1}\, \mathbf{X}^{-1} \tag{C.13}$$

and vice versa as

$$\begin{aligned}
\mathbf{G}_{cl_I}(z) &= \left(\mathbf{I} + \mathbf{N}\,\mathbf{D}^{-1}\,\frac{1}{z-1}\mathbf{K}_I\right)^{-1}\mathbf{N}\,\mathbf{D}^{-1}\,\frac{1}{z-1}\mathbf{K}_I, && \text{(C.14)}\\
&= \left(\mathbf{I}(z-1) + \mathbf{N}\,\mathbf{D}^{-1}\,\mathbf{K}_I\right)^{-1}\mathbf{N}\,\mathbf{D}^{-1}\,\mathbf{K}_I, && \text{(C.15)}\\
&= \left(\mathbf{N}\left(\mathbf{N}^{-1}(z-1) + \mathbf{D}^{-1}\,\mathbf{K}_I\right)\right)^{-1}\mathbf{N}\,\mathbf{D}^{-1}\,\mathbf{K}_I, && \text{(C.16)}\\
&= \left(\mathbf{N}^{-1}(z-1) + \mathbf{D}^{-1}\,\mathbf{K}_I\right)^{-1}\mathbf{N}^{-1}\,\mathbf{N}\,\mathbf{D}^{-1}\,\mathbf{K}_I, && \text{(C.17)}\\
&= \left(\mathbf{N}^{-1}(z-1) + \mathbf{D}^{-1}\,\mathbf{K}_I\right)^{-1}\mathbf{D}^{-1}\,\mathbf{K}_I, && \text{(C.18)}\\
&= \left(\mathbf{D}\left(\mathbf{N}^{-1}(z-1) + \mathbf{D}^{-1}\,\mathbf{K}_I\right)\right)^{-1}\mathbf{K}_I, && \text{(C.19)}\\
&= \left(\mathbf{D}\mathbf{N}^{-1}(z-1) + \mathbf{D}\,\mathbf{D}^{-1}\,\mathbf{K}_I\right)^{-1}\mathbf{K}_I, && \text{(C.20)}\\
&= \left(\mathbf{D}\mathbf{N}^{-1}(z-1) + \mathbf{K}_I\right)^{-1}\mathbf{K}_I, && \text{(C.21)}\\
&= \left(\left(\mathbf{D}\,(z-1) + \mathbf{K}_I\mathbf{N}\right)\mathbf{N}^{-1}\right)^{-1}\mathbf{K}_I, && \text{(C.22)}\\
&= \mathbf{N}\left(\mathbf{D}\,(z-1) + \mathbf{K}_I\mathbf{N}\right)^{-1}\mathbf{K}_I && \text{(C.23)}\\
&= \mathbf{N}\,\mathbf{D}_I^{-1}\,\mathbf{K}_I && \text{(C.24)}
\end{aligned}$$

with

$$\mathbf{D}_I = \mathbf{D}\,(z-1) + \mathbf{K}_I\mathbf{N}. \tag{C.25}$$

C.3 Controller computation for PI-structure

Similarly to [Nazaruddin, 1994] the controller computation for PI-structure is derived in the following for the rotor bearing system.

The desired polynomial matrix for the closed loop system with integrative feedback is

$$\mathbf{P}_I = \begin{bmatrix} z^3 + p_{I_1}^{(11)} z^2 + p_{I_2}^{(11)} z + p_{I_3}^{(11)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & z^3 + p_{I_1}^{(44)} z^2 + p_{I_2}^{(44)} z + p_{I_3}^{(44)} \end{bmatrix}. \tag{C.26}$$

After the matrix gain has been found as

$$\mathbf{K}_I = \mathbf{P}_I(z)\,\mathbf{N}^{-1}(z)\Big|_{z=1} \tag{C.27}$$

with the controller gain matrix for the integrative feedback

$$\mathbf{K}_I = \begin{bmatrix} k_I^{(11)} & \dots & k_I^{(14)} \\ \vdots & \ddots & \vdots \\ k_I^{(41)} & \dots & k_I^{(44)} \end{bmatrix}. \tag{C.28}$$

The equation for the polynomial matrix $\mathbf{D}$ has to be solved as

$$\mathbf{P}_I \stackrel{!}{=} \mathbf{D}_I = (z-1)\,\mathbf{D} + \mathbf{K}_I\,\mathbf{N}. \tag{C.29}$$

Using the previous definitions for all matrices, the diagonal entries of the closed loop polynomial matrix collected in positive powers of z are

$$\begin{aligned} \mathbf{D}_I^{(ii)} &= z^3 + \left(p_1^{(ii)} - 1\right) z^2 \\ &\quad + \left(\sum_{j=1}^{4} c_2^{ji}\, k_w^{(ij)} - p_1^{(ii)} + p_2^{(ii)}\right) z \\ &\quad + \left(\sum_{j=1}^{4} c_2^{ji}\, k_w^{(ij)} - p_2^{(ii)}\right) \\ &= z^3 + p_{I_1}^{(ii)} z^2 + p_{I_2}^{(ii)} z + p_{I_3}^{(ii)} \end{aligned} \tag{C.30}$$

Comparing these diagonal polynomials to the desired ones in $\mathbf{P}_I$, yields

$$p_1^{ii} = p_{I_1}^{ii} + 1, \tag{C.31}$$

$$p_2^{ii} = \sum_{j=1}^{4} c_2^{ji}\, k_w^{(ij)} - p_{I_3}^{ii}. \tag{C.32}$$

Note that the solution is unique, although only two parameters were calculated from three equations. If the solution is derived from the equation with z^2 and z^0 the expression with z^1 turns out to be redundant. The non-diagonal elements in $\mathbf{D}_I$ are zero according to Eqn.(C.27).